magnitude over the altitude range of interest. But with the currently available technology in deployable and retractable structures it should not be too difficult to vary the control panel areas by orders of magnitude. Such variable area control panels will lead to better performance at higher altitudes.

Conclusions

The performance characteristics and capabilities of various satellite attitude control systems have been compared in Sabroff's paper.[8] Comparison of the present system with other systems on the basis of the various characteristics enunciated[8] is possible only if a fairly complex simulation of the over-all system is carried out. Since this simulation has not been done, only the general range of performance of the present system will be indicated. In the present case stabilization is not possible about an arbitrary attitude whereas it may be possible for mass expulsion systems. Initial attitude angles from which acquisition is possible are fairly large, but the allowable intial rates are relatively small. This compares favorably with most systems except momentum storage/mass expulsion systems. The present system requires attitude sensing and attitude rate information about all three axes. Because of the feedback control scheme utilized, the range of orientation accuracy is expected to be reasonably good. Control cost involves only the power required to rotate the control panels and does not involve any fuel expenditure.

The results of this study indicate that it is possible to exploit the aerodynamic forces acting on a satellite orbiting at certain altitudes to actively control the attitude in an Earth-pointing mode. The density variations due to the diurnal bulge have been shown to be of no serious consequence in devising a suitable feedback control system. Further it has been shown that the extreme variations in density with altitude can be handled by either a gain scheduling scheme or by providing a set of control panels whose areas are varied suitably with altitude. Numerical results indicate that the effect of off-design values of surface accommodation coefficients do not cause any serious problems. Variable gain feedback matrices for nonequatorial orbits have been shown to be unnecessary An evaluation of the effect of the panels on orbit lifetime has been carried out and the results indicate that lifetimes of the order of two to three years are possible.[6]

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Absolute Stability Analysis of Attitude Control Systems for Large Boosters

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A method for performing absolute stability analyses of attitude control systems for large launch vehicles is presented. Absolute stability of these systems is shown in a finite region of the state space. The regions are computed by using the Lur'e-Postnikov Liapunov function. This function is chosen to provide additional information about the exponential property of absolute stability. Significant advantages of the method proposed in this paper are: It is independent of the order of the system; algebraic operations involved in the computations are relatively simple and convenient for machine implementation; and the obtained results are valid not only for a particular nonlinearity but also for an entire class of nonlinear characteristics that satisfy certain general conditions. A system model representing the Saturn V launch vehicle is used to illustrate the method.

Introduction

It will be shown how the attitude control systems for large boosters can be analyzed within the framework of absolute stability. The approach to be developed in this paper opens new avenues for the application of numerous strong results of absolute stability theory[1] to the control of large launch vehicles.

A saturation-type characteristic is used to represent the hydraulic actuators that rotate the gimbaled engines of the boosters. Since the linear part of the system is not stable, the saturation characteristic violates the sector condition, and the absolute stability of the attitude control system can be shown to lie in a finite region of the state space. Regions of absolute stability are estimated by using Liapunov functions of the Lur'e-Postnikov type. An extended version[2,3] of the Popov frequency criterion and the Yakubovich matrix inequalities is used in selecting an appropriate Liapunov function which

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provides additional information about the exponential property of absolute stability. This property guarantees a desired degree of rapidity of the transient process that takes place in the system when perturbations occur.

A method for estimating the regions of exponential absolute stability was introduced in Ref. 4 where only the quadratic form was used as a Liapunov function. This paper utilizes the Lur'e-Postnikov function, "quadratic form plus an integral of the nonlinearity," which may yield improvements in the estimates of the regions at the expense of losing the quadratic property of the estimates. Moreover, in satisfying the extended version of the Popov frequency condition, stronger requirements are placed on the linear part of the system. Once the Liapunov function of the Lur'e-Postnikov type is established by using the frequency condition, the regions may be calculated as proposed in Refs. 5 and 6.

Liapunov instability criteria and frequency domain techniques were used[7] for selecting the feedback control parameters in an attitude control system so that a chosen initial state is excluded from the predicted region of instability. Conservative results were obtained so that insufficient information was provided for the estimation of parameters and states that assure system stability. In this direction, considerations of attitude control systems for large launch vehicles by approximate methods[8] are more promising because they provide stronger "necessary conditions" in the estimation of stability regions in both the parameter and the state space. The approximate methods can be supplemented with the analysis of parametric absolute stability[1,9] to yield a useful addendum to the method presented herein.

Absolute Stability Regions

Attitude control systems for large boosters can be cast in the form of the Lur'e-Postnikov class of nonlinear systems which are described by‡

$$\dot{x}(t) = Px(t) + q\phi(\sigma), \quad \sigma(t) = r^T x(t) \tag{1}$$

In Eqs. (1), $x(t)$ is an n vector—the state of the system; P is a constant $n \times n$ matrix; q and r are constant n vectors; and $\phi(\sigma)$ is a scalar, a continuous function, or a discontinuous function with only isolated points of discontinuity of the first kind.

In the absolute stability analysis of the attitude control system, the following aspects are essential. a) For absolute stability of system (1), global asymptotic stability of the equilibrium $x = 0$ is required for all nonlinear characteristics $\phi(\sigma)$ which belong to the class defined as

$$\Phi_\kappa: 0 \leq \sigma\phi(\sigma) \leq \kappa\sigma^3, \quad \kappa \leq \infty$$
$$\phi(0) = 0 \tag{2}$$

Therefore, the linear part of system (1), described by the nondegenerate transfer function

$$\chi(\lambda) = r^T(P - \lambda I)^{-1} \tag{3}$$

should be asymptotically stable, i.e., the matrix P in Eqs. (1) should be Hurwitz. The vehicle dynamics give rise to zero eigenvalues of P. Hence, a transformation[1] of Eqs. (1) is necessary to achieve stability of the linear part. b) The function $\phi(\sigma)$, which represents the nonlinearity of the system, is generally a saturation-type characteristic. After the transformation mentioned above, $\phi(\sigma)$ violates the inequality in Eq. (2). Therefore, absolute stability can be shown in a finite region of the state space.[4,6] c) In the analysis of an attitude control system, merely knowing that the system is stable is not sufficient. It is also important to provide information about the rapidity of the transient process that takes place in the system after possible perturbations. A common comparison function is the exponential function. It will be shown that there exist two positive constants (ρ, δ) independent of the initial values (x_0, t_0) such that the system motion $x(t; x_0, t_0)$ satisfies the inequality

$$\|x(t; x_0, t_0)\| \leq \rho \|x_0\| e^{-\delta(t-t_0)} \tag{4}$$

for all x_0, t_0, $t > t_0$, and any $\phi(\sigma)\epsilon\Phi_\kappa$. With this property, the system in Eqs. (1) is said to be exponentially absolutely stable.[1-4] To achieve this kind of stability, the transformation previously mentioned must be aimed at making the matrix $P + \delta I$ Hurwitz. This, in turn, will limit the exponential property of absolute stability to a finite extent.

The absolute stability determination of the system in Eqs. (1) is begun by adding to the right-hand side the zero identity $\kappa_\delta q(r^T x - \sigma) \equiv 0$, so that Eqs. (1) become

$$\dot{x}(t) = P_{tr}x(t) + q\phi_{tr}(\sigma), \quad \sigma(t) = r^T x(t) \tag{5}$$

where

$$P_{tr} = P + \kappa_\delta q r^T, \quad \phi_{tr}(\sigma) = \phi(\sigma) - \kappa_\delta \sigma \tag{6}$$

The constant κ_δ is chosen as a number that makes the matrix

$$P_\delta = P_{tr} + \delta I \tag{7}$$

a Hurwitz matrix. This is a linear problem, and after a desirable number is chosen for δ any of the linear methods (e.g., parameter method, root locus, etc.) can be used to select an appropriate value for κ_δ so that all of the poles of

$$\chi_{tr}(\lambda) = \chi(\lambda)/[1 + \kappa_\delta \chi(\lambda)] \tag{8}$$

are located inside the half-plane $Re\,\lambda < -\delta \leq 0$.

Once a value of κ_δ is found, one applies the absolute stability analysis to Eqs. (5). The exponential property of absolute stability of Eqs. (5) is guaranteed by satisfying the extended version of the Popov inequality

$$\pi(\omega) \equiv \kappa_{tr}^{-1} + Re\{(1 + j\theta\omega)\chi_{tr}(-\delta + j\omega)\}$$
$$- \theta\delta\kappa_{tr}|\chi_{tr}(-\delta + j\omega)| > 0 \tag{9}$$

for some real θ, all real $\omega \geq 0$, and $\kappa_{tr} \geq \kappa - \kappa_\delta \geq 0$.

As shown in Ref. 3, condition (9) is necessary and sufficient for the existence of a Liapunov function

$$V(x) = x^T H x + \theta \int_0^{r^T x} \phi(\sigma) d\sigma, \quad H = H^T \tag{10}$$

having the derivative $\dot{V}$ along the solutions of Eqs. (5) such that

$$-\dot{V} - 2\delta V > [x^T G_\delta x + 2\gamma^{1/2}\phi x^T g_\delta + \gamma\phi^2] + (\sigma - \phi\kappa_{tr}^{-1})\phi$$
$$+ \theta\delta\phi\sigma \tag{11}$$

where

$$-G_\delta = HP_\delta + P_\delta{}^T H + \theta\delta\kappa_{tr} r r^T$$
$$-(\gamma)^{1/2} g_\delta = Hq + \tfrac{1}{2}(\theta P_\delta{}^T + I)r$$
$$-\gamma = \theta r^T q - \kappa_{tr}^{-1} \tag{12}$$

To obtain Eq. (11) after the differentiation of V, the zero identity $\theta\delta\kappa_{tr}(\sigma^2 - x^T r r^T x) \equiv 0$ is added to the expression $-\dot{V} - 2\delta V$ and use is made of the inequality

$$\tfrac{1}{2}\kappa_{tr}\sigma^2 - \int_0^\sigma \phi(\sigma) d\sigma \geq 0, \quad \text{for all } \sigma \tag{13}$$

which follows from the sector inequality in Eq. (2).

‡ Capital Roman letters denote matrices, lower case Roman letters denote vectors, capital Greek letters denote sets, and lower case Greek letters denote scalars. The letter t is used only for time, and the letter V only for a Liapunov function. Vectors are considered as column matrices, and superscript T denotes the transpose. The notation $H > 0$ means that H is a positive definite matrix, and I is the identity matrix.

The matrix $H > 0$ satisfies the matrix inequalities

$$G_\delta - g_\delta g_\delta^T > 0, \quad \gamma \neq \infty$$
$$G_\delta > 0, \quad g_\delta = 0, \quad \gamma = \infty \tag{14}$$

For $\theta \geq 0$,

$$x^T H x \leq V(x) \leq x^T H_1 x \tag{15}$$

where $H_1 = H + \frac{1}{2}\theta\kappa_{tr} rr^T$ and $H > 0$. Therefore, from Eqs. (11) and (15), it is concluded that it is always possible to find a sufficiently small positive number $\epsilon > 0$ such that

$$-\dot{V} - 2\delta V \geq 2\epsilon V \tag{16}$$

Separating the variables and integrating with respect to time, one obtains the inequality

$$V[x(t;\, x_0,\, t_0)] \leq V(x_0)e^{-2(\delta+\epsilon)(t-t_0)} \tag{17}$$

which is valid for all $t \geq t_0$.

From Eq. (15), one obtains

$$\mu\|x\|^2 \leq V(x) \leq \mu_1\|x\|^2 \tag{18}$$

where μ and μ_1 are minimum and maximum eigenvalues of H and H_1, respectively. From inequality (18) follows the inequality (4) with $\rho = (\mu_1/\mu)^{1/2}$ and δ replaced by $\delta + \epsilon$. Thus, exponential stability of the system (5) is established for $\theta \geq 0$. If frequency condition (9) is fulfilled for $\theta < 0$, a transformation in Ref. 1, p. 552, can be used to obtain the same result.

Let us now consider a saturation-type characteristic

$$\phi(\sigma) = \begin{cases} \xi\sigma, & |\sigma| \leq \eta \\ \xi\eta \operatorname{sign}\sigma, & |\sigma| \geq \eta \end{cases} \tag{19}$$

where ξ, η are positive constants. Because of the transformation in Eqs. (6), the characteristic $\phi_{tr}(\sigma)$ violates the condition

$$\sigma\phi_{tr}(\sigma) \geq 0 \text{ for } |\sigma| > \alpha, \quad \alpha = \xi\eta\kappa_\delta^{-1} \tag{20}$$

which is necessary for absolute stability of Eqs. (5). Therefore, we enlarge the class of nonlinear functions and require that $\phi_{tr}(\sigma)$ belong to

$$\Phi_{\kappa_{tr},\alpha}\colon 0 \leq \sigma\phi_{tr}(\sigma) \leq \kappa_{tr}\sigma^2, \quad |\sigma| < \alpha$$
$$\phi_{tr}(0) = 0 \tag{21}$$

but, in turn, limit the analysis of exponential absolute stability to a finite region in the state space.[4]

The region Ω_δ of exponential absolute stability is defined as the set of all points x_0 for which the solutions $x(t;\, x_0,\, t_0)$ of Eqs. (1) starting at x_0 are exponentially absolutely stable; i.e., exponentially stable for any $\phi_{tr}(\sigma)\epsilon\Phi_{\kappa_{tr},\alpha}$.

To make the class $\Phi_{\kappa_{tr},\alpha}$ of nonlinear functions $\phi_{tr}(\sigma)$ as large as possible, one applies the Popov graphical construction to inequality (9) and finds the maximum value of κ_{tr} for which that inequality is satisfied. If the constant κ_{tr} is known, as in the case with nonlinearity (19), one can use the simple algebraic criterion for absolute stability[10] and verify the inequality by a purely numerical algorithm convenient for machine applications.

Once the constant κ_{tr} is determined and a given $\phi_{tr}(\sigma)$ is such that $\phi_{tr}(\sigma)\epsilon\Phi_{\kappa_{tr},\alpha}$ for some α, one is interested in finding the largest region Ω_δ^m ($\Omega_\delta^m \subset \Omega_\delta$) defined by

$$\Omega_\delta^m = \{x\colon V(x) < \beta\} \tag{22}$$

where the number β is determined from

$$\beta = \min_{|\sigma(x)| = \alpha} V(x), \tag{23}$$

To calculate β, one may proceed as proposed in Ref. 5. From Eqs. (22) and (23) one obtains

$$2Hx + r\nu = -\theta\phi r$$
$$r^T x = \alpha \tag{24}$$

The solution of linear Eqs. (24) for x and the constant ν yields

$$\beta = \alpha^2(r^T H^{-1} r)^{-1} + \theta\int_0^\alpha \phi(\sigma)d\sigma \tag{25}$$

If, however, one wants the region Ω_δ^m to be valid for all $\phi_{tr}\epsilon\Phi_{\kappa_{tr},\alpha}$ one uses

$$\Omega_\delta^m = \{x\colon x^T H x < \beta_1\}, \quad \text{for } \theta \geq 0 \tag{26}$$

where

$$\beta_1 = \alpha^2[(r^T H^{-1} r)^{-1} + \tfrac{1}{2}\theta\kappa_{tr}] \tag{27}$$

or

$$\Omega_\delta^m = \{x\colon x^T H_1 x < \beta_2\}, \text{ for } \theta < 0 \tag{28}$$

where

$$\beta_2 = \alpha^2(r^T H^{-1} r)^{-1} \tag{29}$$

As suggested in Ref. 3, one can use a modification of the Kalman construction to determine a specific H from

$$HP_\delta + P_\delta^T H = -uu^T - \theta\delta\kappa_{tr} rr^T \tag{30}$$

if $\pi(\omega)$ in equality (9) is factored as

$$\pi(\omega) = \xi(j\omega)\xi(-j\omega)/|P_\delta - j\omega I|\,|P_\delta + j\omega I| \tag{31}$$

and the vector u is chosen such that

$$u^T(P_\delta - \lambda I)^{-1}q = [\xi(\lambda)/|P_\delta - \lambda I|] - \gamma^{1/2} \tag{32}$$

Note that a different matrix H produces a different region Ω_δ^m with respect to the extent and orientation, and it is desirable to select one that provides in some sense the best estimate $\tilde{\Omega}_\delta^m$ of the region of exponential absolute stability. Then, a set of H matrices can be generated directly from the matrix inequalities (14), which can be reduced to Sylvester inequalities involving the elements of H. The best estimate $\tilde{\Omega}_\delta^m$ may be determined, for example, by maximizing the volume of Ω_δ^m over the set of generated H matrices, which is a simple matter for quadratic regions.

As shown by a simple example in Ref. 4, the size of the region Ω_δ^m is a function of δ. The larger δ is chosen, the smaller Ω_δ^m becomes. The number δ may be systematically varied to establish a satisfactory tradeoff between the degree of stability and extent of the region.

From the above development, one can expect that the results for a particular nonlinearity are not likely to be as good an estimate as those found by a systematic and effective procedure such as that of Ref. 11 or by computer simulation. However, by the proposed method a quadratic region can be made valid for all nonlinear characteristics $\phi_{tr}(\sigma)\epsilon\Phi_{\kappa_{tr},\alpha}$; this is of considerable interest in applications where the nonlinearity is not known precisely or may vary in time. In addition, the complexity of the proposed algorithm does not increase substantially with the order of the system—a feature absent in computer simulation. Furthermore, the exponential property of absolute stability cannot be practically determined from a computer simulation.

In concluding this section, note that the above development for $\theta = 0$ becomes that of Ref. 4, and the $\delta = 0$ case becomes that of Ref. 5.

Large Booster Attitude Control System Description

A typical attitude control system for a large booster is the Saturn launch vehicle described in Ref. 12. For simplicity, it is assumed that the vehicle under consideration is rigid and flying above the sensible atmosphere. If rotational motion is

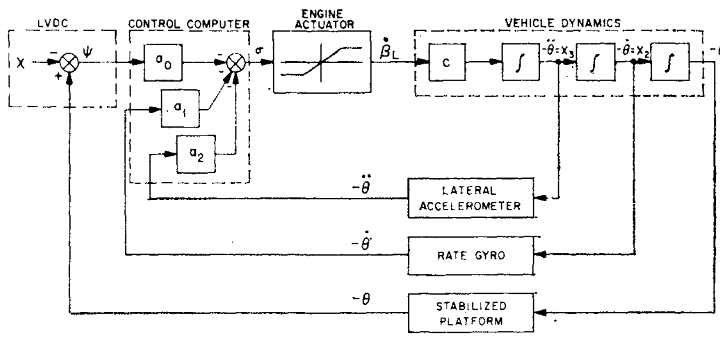

Fig. 1 Simplified attitude control system.

considered in only the pitch plane, rotational rigid-body dynamics may be described by

$$\ddot{\theta} + c\beta_L = 0 \tag{33}$$

where

$$c = l(\partial R/\partial \beta_L)/I \tag{34}$$

The term l represents the distance between the engine gimbal point and the vehicle center of mass, R represents the lateral component of engine thrust, β_L is the angle in the pitch plane between the thrust vector and the longitudinal axis of the vehicle, and I represents the vehicle's moment of inertia about the pitch axis. The symbol θ represents the pitch attitude of the vehicle.

Numerous attitude control laws have been postulated for launch vehicles and used successfully. One postulate may be written as

$$\sigma = a_0\Psi + a_1\dot{\theta} + a_2\ddot{\theta} \tag{35}$$

where σ represents the commanded thrust vector angular rate; Ψ is the measured attitude error; and a_0, a_1, and a_2 are adjustable control gains. If it is assumed that the engine actuator produces a thrust vector angular rate proportional to the commanded rate until a saturation limit (η) is reached, the actuator may be modeled as Eq. (19), where $\phi(\sigma)$ represents $\dot{\beta}_L$.

With the use of Eqs. (33) and (34), the attitude control system of Ref. 12 may be simplified so that it appears as shown in Fig. 1. For additional simplicity, it is assumed that the output of the lateral accelerometer is angular acceleration.

Application

Consider the simplified system model for the attitude control of a large booster given in Fig. 1. If a typical set of numerical values is chosen for c, a_0, a_1, a_2, ξ, and η as 2, 2, 1, 1.25, 10, and 2, respectively, the model is described by Eqs. (1) where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \quad q = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}; \quad r = \begin{bmatrix} 4 \\ 2 \\ 2.5 \end{bmatrix} \tag{36}$$

and

$$\phi(\sigma) = \begin{cases} 10\sigma, & |\sigma| \leq 2 \\ 20\ \mathrm{sign}\sigma, & |\sigma| \geq 2 \end{cases} \tag{37}$$

This simple model, however, has all the characteristic properties of the considered class of systems necessary for a meaningful illustration of the outlined method of analysis. Since the method does not depend on the order of the system, it can be applied directly to more complex situations.

The transfer function in Eq. (3) of the linear part of the system is

$$\chi(\lambda) = (2.5\lambda^2 + 2\lambda + 4)/\lambda^3 \tag{38}$$

By applying a linear analysis, one can find that for $\kappa_{0.2} = 1.7$, all the poles of the transfer function in Eq. (8) computed as

$$\chi_{tr}(\lambda) = \frac{2.5\lambda^2 + 2\lambda + 4}{\lambda^3 + 4.25\lambda^2 + 3.4\lambda + 6.8} \tag{39}$$

are located inside the half-plane $Re\ \lambda < -0.2$.

According to the transformation in Eq. (6), the system to be analyzed is specified by

$$P_{tr} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6.8 & -3.4 & -4.25 \end{bmatrix}; \quad q = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}; \quad r = \begin{bmatrix} 4 \\ 2 \\ 2.5 \end{bmatrix} \tag{40}$$

To simplify the illustration of the proposed method, it is assumed that $\theta = 0$. However, note that improved results can be obtained when $\theta \neq 0$, but the estimates of the regions of exponential absolute stability are no longer quadratic.

Applying the Popov graphical test to inequality (9), the Popov diagram of Fig. 2 is obtained for $\chi_{tr}(\lambda)$ as in Eq. (39). As is clear from Fig. 2, one can choose $\kappa = \infty$. Consequently, the system specified by Eq. (40) is stable exponentially in the entire sector $[0, \infty]$. Because of the transformation $\phi_{tr}(\sigma) - 1.7\sigma$, the transformed saturation nonlinearity violates the sector $[0, \infty]$ at $|\sigma| = \alpha = 11.6$ as shown in Fig. 3. Therefore, all functions $\phi_{tr}(\sigma)\epsilon\Phi_{\infty,11.6}$ that lie in the shaded region of Fig. 3 are considered.

Now to determine the region $\Omega_\delta{}^m$ of exponential absolute stability for $\theta = 0$, $\phi_{tr}(\sigma)\epsilon\Phi_{\infty,11.6}$, and $\chi_{tr}(\lambda)$ given in Eq. (39), one takes $\kappa = \infty$ and factors the corresponding expression $\pi(\omega)$ of inequality (9) as shown in Eq. (31). This operation yields the vector $u = [-0.22, -0.74, -2.85]^T$ and the matrix

$$H = \begin{bmatrix} 628.46 & 242.26 & 184.88 \\ 242.26 & 444.02 & 974.51 \\ 184.88 & 974.51 & 250.65 \end{bmatrix} \tag{41}$$

as specified by Eqs. (30) and (32). Substituting this H in Eq. (25), one obtains $\beta = 0.44$ as defined in Eq. (23). Finally, one obtains an estimate of the region of exponential absolute stability for the system under investigation as

$$\Omega_{0.2}{}^m = \{x: x^T H x < 0.44\} \tag{42}$$

where H is the matrix specified in Eq. (41).

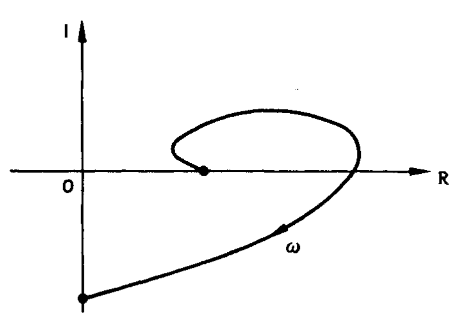

Fig. 2 Popov diagram.

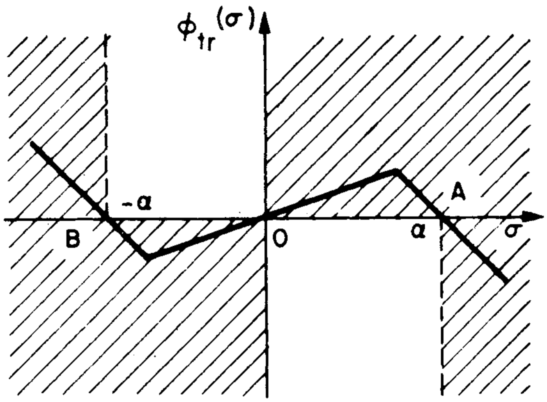

Fig. 3 Augmented nonlinearity sector.

It is of interest to note that the computed region $\Omega_{0.2}{}^m$ in Eq. (42) corresponds to all $\phi_{tr}(\sigma)\epsilon\Phi_{\infty,11.6}$, which is the consequence of absolute stability. If one notes that the transformed system is exponentially absolutely stable in the sector $[0, \infty]$, which implies that the original system has the same properties of stability for any nonlinearity that belongs to the sector $[1.7, \infty]$, one concludes that the relay characteristic $\phi(\sigma) = \text{sign}\sigma$ can also be used in the system in Eqs. (1) without affecting the region $\Omega_{0.2}{}^m$ given in Eq. (42). The relay characteristic, being a discontinuous function of the first kind, may produce sliding motions. These kinds of absolute stability are shown in detail in Ref. 1, and no additional analysis is necessary after the region $\Omega_\delta{}^m$ is computed.

The value of infinity was chosen for κ to simplify the calculations in the example. However, it is reasonable to expect that better results in terms of a larger region of stability would accrue if a smaller value of κ were chosen, as is clear from Fig. 2. In other words, a tradeoff probably exists between the size of the region of stability and the upper limit of the nonlinear sector. Of course, the upper limit cannot be less than the slope of the nonlinearity under consideration. In practice, a factor of safety would be included by insuring that κ is selected so that the Popov sector always includes the nonlinearity under study, even under the influence of expected perturbations.

Conclusion

It has been shown how the regions of exponential absolute stability in the state space can be computed for attitude control systems for large boosters. The Lur'e-Postnikov type of Liapunov function was used to estimate these regions. The results obtained by applying the proposed method may appear exceedingly conservative when compared with methods based upon the precise description of a particular relevant nonlinear characteristic. However, a strict comparison of the obtained result is improper, since there are four unique advantages of this method. a) The algebra is simple and applies to systems of high order, a feature absent in other approaches, in which the algebraic difficulty significantly increases with system order. b) Results may be obtained for a class of nonlinearities which may not be known precisely and may even vary in time. c) The stability regions may be found as an explicit function of α, where the nonlinearity violates the sector condition. d) Additional information is provided about the rapidity of the transient process that takes place in the system.

It is of interest to note that the property of exponential stability makes it possible to consider perturbations on the system and determine a bound on the forcing function that guarantees that all the solutions remain bounded inside the computed region. This problem is solved in Ref. 4.

Additional freedom can be provided by the proper choice of the regulator vector r as shown in Ref. 9. The problem of selecting a regulator vector which yields the largest estimate of the region of exponential absolute stability can be recast as a problem of maximizing the volume of the estimate by using mathematical programming methods. Using the same example as described in this paper, distinct improvements of the estimate are accomplished, and the results are tabulated in Ref. 9.

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